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Conserved superenergy currents

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Abstract

We exploit once again the analogy between the energy-momentum tensor and the so-called "superenergy" tensors in order to build conserved currents in the presence of Killing vectors. First of all, we derive the divergence-free property of the gravitational superenergy currents under very general circumstances, even if the superenergy tensor is not divergence-free itself. The associated conserved quantities are explicitly computed for the Reissner-Nordström and Schwarzschild solutions. The remaining cases, when the above currents are not conserved, lead to the possibility of an interchange of some superenergy quantities between the gravitational and other physical fields in such a manner that the total, *mixed*, current may be conserved. Actually, this possibility has been recently proved to hold for the Einstein-Klein-Gordon system of field equations. By using an adequate family of known exact solutions, we present explicit and completely non-obvious examples of such mixed conserved currents.

1 Introduction and basic results

Conserved quantities in Gravitation are either defined only at "infinity", see e.g. [33] and references therein, or they arise as a consequence of the existence of divergence-free vector fields, called *local currents*, and of the use of Gauss' theorem applied to appropriate domains of the spacetime, see e.g. [13, 22, 26]. A traditional way of constructing divergence-free vector fields is by using the intrinsic symmetries of the spacetime[1]: if $\vec{\xi}$ is a Killing vector field, $\nabla_{(\alpha}\xi_{\beta)} = 0$, representing an infinitesimal isometry, then many local currents and associated conserved quantities arise, such as for instance every vector field of the form $A\vec{\xi}$, where A is any function

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[1] A spacetime $(\mathcal{V}, g)$ is a 4-dimensional manifold $\mathcal{V}$ with a metric g of Lorentzian signature (−,+,+,+). Indices in $\mathcal{V}$ run from 0 to 3 and are denoted by Greek small letters. Square and round brackets denote the usual (anti-) symmetrisation of indices. The tensor and exterior products are denoted by $\otimes$ and $\wedge$, respectively, boldface letters are used for 1-forms and arrowed symbols for vectors, and the exterior differential is denoted by d. The Lie derivative with respect to the vector field $\vec{\xi}$ is written as $\pounds_{\vec{\xi}}$, and the covariant derivative by ∇. Equalities by definition are denoted by $\equiv$, and the end of a proof is signalled by $\blacksquare$.

satisfying $\vec{\xi}(A) = 0$. This is a simple consequence of the invariance of the geometrical background under the Killing vector field. Of course, if there are several Killing vectors $\vec{\xi}_L$ in the spacetime, then any linear combination of the vector fields of type $A^L\vec{\xi}_L$ is also divergence-free whenever each A^L is invariant with respect to the corresponding $\vec{\xi}_L$.

There are also other, physically more interesting, types of locally conserved currents constructible from $\vec{\xi}$ together with other objects. Take, for example, the typical case of $t^{\mu\nu}\xi_\nu$ for a *divergence-free symmetric* tensor $t^{\mu\nu}$. Obviously we have

$$\nabla_\mu \left(t^{\mu\nu}\xi_\nu\right) = \left(\nabla_\mu t^{\mu\nu}\right)\xi_\nu + t^{[\mu\nu]}\nabla_\mu \xi_\nu = 0$$

where the Killing property $\nabla_{(\alpha}\xi_{\beta)} = 0$ has been used. Observe that both the divergence-free property *and* the symmetry of $t^{\mu\nu}$ are needed here. In fact the divergencelessness of $t^{\mu\nu}$ is a sufficient but not necessary condition, as one only needs that $\vec{\xi}$ and $\nabla \cdot t$ be orthogonal for the first term to vanish. Furthermore, observe that just a *conformal* Killing vector field ($\nabla_{(\alpha}\xi_{\beta)} \propto g_{\alpha\beta}$) is required if $t_{\mu\nu}$ is also traceless.

A particular important example of this type of locally conserved currents is provided by the case of the total energy-momentum tensor $T^{\mu\nu}$ of the matter fields in General Relativity. In this case, due to the Einstein field equations, $T^{\mu\nu}$ is symmetric and divergenceless, and thus any Killing vector field $\vec{\xi}$ provides a divergence-free current defined by

$$\mathcal{J}^\alpha\left(\vec{\xi}\right) \equiv T^{\alpha\beta}\xi_\beta. \tag{1}$$

These currents have also a physical meaning which depends on the particular character of the Killing vector $\vec{\xi}$. For instance, in stationary spacetimes there is a timelike Killing vector $\vec{\xi}$, and the corresponding current $\vec{\mathcal{J}}\left(\vec{\xi}\right)$ can be thought of as the local energy-momentum vector of the matter in the stationary system of reference. Similarly, for translational or rotational Killing vectors one obtains currents representing the whole linear or angular momentum, respectively, at least in flat spacetimes.

Two important remarks are in order here. First, in many cases the mentioned currents $\vec{\mathcal{J}}\left(\vec{\xi}\right)$ are of the form referred to above, that is, $\vec{\mathcal{J}}\left(\vec{\xi}\right) = B^L\vec{\xi}_L$ for some particular functions B^L. This does not deprive $\vec{\mathcal{J}}\left(\vec{\xi}\right)$ of its physical significance. Rather one has to see this result as stating that, among the huge variety of divergence-free vectors of the form $A^L\vec{\xi}_L$ with general A^L, the particular one with the functions $A^L = B^L$ have a specially relevant physical meaning. Several cases in which this happens are given by the following known result [18, 23, 9, 21, 16, 31].[2]

Result 1.1 *Assume Einstein's field equations of General Relativity hold.*

1. *If $\vec{\xi}$ is a hypersurface-orthogonal Killing vector, then* $\boldsymbol{\mathcal{J}}\left(\vec{\xi}\right) \wedge \boldsymbol{\xi} = 0$.

2. *If $\vec{\xi}$ and $\vec{\eta}$ are two commuting Killing vectors $[\vec{\xi}, \vec{\eta}] = 0$ and they act orthogonally transitively on non-null surfaces, then* $\boldsymbol{\mathcal{J}}\left(\vec{\xi}\right) \wedge \boldsymbol{\xi} \wedge \boldsymbol{\eta} = \boldsymbol{\mathcal{J}}\left(\vec{\eta}\right) \wedge \boldsymbol{\xi} \wedge \boldsymbol{\eta} = 0$.

[2] Recall that a vector field $\vec{v}$ is called hypersurface-orthogonal (also "integrable") if $\boldsymbol{v} \wedge \mathrm{d}\boldsymbol{v} = 0$, and that two vector fields orthogonal to the non-null surfaces spanned by two given vector fields $\vec{v}$ and $\vec{w}$ generate surfaces whenever the two 1-forms $\boldsymbol{v}$ and $\boldsymbol{w}$ satisfy $\boldsymbol{v} \wedge \boldsymbol{w} \wedge \mathrm{d}\boldsymbol{w} = \boldsymbol{v} \wedge \boldsymbol{w} \wedge \mathrm{d}\boldsymbol{v} = 0$. Thus a G_2-group of motions generated by two Killing vectors $\vec{\xi}$ and $\vec{\eta}$ is said to be acting orthogonally transitively on non-null orbits when there exists a family of surfaces orthogonal to the orbits, which means that $\boldsymbol{\xi} \wedge \boldsymbol{\eta} \wedge \mathrm{d}\boldsymbol{\eta} = \boldsymbol{\xi} \wedge \boldsymbol{\eta} \wedge \mathrm{d}\boldsymbol{\xi} = 0$.

The first of these results means that if there are (locally) hypersurfaces orthogonal to the Killing vector, then the corresponding current lies in the direction of the Killing vector itself. This implies that the integration of $\vec{\mathcal{J}}\left(\vec{\xi}\right)\cdot\vec{\xi}$ (and of $\vec{\mathcal{J}}\left(\vec{\xi}\right)\cdot\vec{\xi}/|\vec{\xi}|$) over any two different hypersurfaces orthogonal to $\vec{\xi}$ is the same. The second one states that any two (group-forming) Killing vectors whose surfaces of transitivity are orthogonal to a family of surfaces have currents which lie necessarily in the orbits of the G_2 group generated by them. Of course, both results are quite natural and follow from the invertibility of the orthogonally transitive Abelian G_2 group (which reduces to hypersurface-orthogonality for 1-dim groups) [9].

The second remark concerns the case when there are several types of matter fields in the spacetime, and the corresponding energy-momentum tensors. For simplicity, assume that there are two different sorts of matter contents (an electromagnetic field and a fluid for instance), with corresponding energy-momentum tensors $T^{(1)}_{\alpha\beta}$ and $T^{(2)}_{\alpha\beta}$, forming an isolated system. Then, only the *total* energy-momentum tensor $T_{\alpha\beta}\equiv T^{(1)}_{\alpha\beta}+T^{(2)}_{\alpha\beta}$ is divergence-free, and therefore the partial currents constructed from $T^{(1)}_{\alpha\beta}$ or $T^{(2)}_{\alpha\beta}$ are not conserved separately in general. Of course, this is physically meaningful and reasonable, as it leads to the conservation of the corresponding total quantity, and thereby implies the *exchange* of some energy-momentum properties between the two fields involved. A classical example of this situation can be found in the standard reference [19] for the case of Special Relativity, where one can see that if the sources of the electromagnetic field are taken into account then the energy-momentum tensor of the electromagnetic field is not divergence-free, but its combination with the energy-momentum tensor of the charges creating the field provides a total energy-momentum tensor which is divergence-free and thereby provides the usual conserved currents associated to the ten Killing vectors of flat spacetime. This proves in particular that the energy-momentum properties can be transferred from one field to another, or from matter to the fields, and vice versa. This transference property together with the positivity and the conservation are the basis for the paramount importance of the concept of energy-momentum in physical theories.

In this paper we shall analyse other locally conserved currents arising in Gravitation which keep the mentioned three properties: positivity, local conservation, and interchange. These local currents arise due to the existence of the so-called "superenergy" tensors, see [1, 2, 3, 4, 25, 27, 28] and references therein, and, as we shall prove, they have analogous attributes to those so far mentioned for the energy-momentum currents (1). The type of currents we shall analyse have expressions such as

$$T^{\alpha\beta\lambda\mu}\,\xi_\beta\xi_\lambda\xi_\mu,$$

or adequate generalisations of this using different Killing vectors, where $T^{\alpha\beta\lambda\mu}$ will be any of the available superenergy tensors for the physical fields involved.

In particular, we start by constructing the corresponding currents for the Bel superenergy (s-e) tensor [2, 3, 28], which is divergence-free in vacuum (where it is called the Bel-Robinson tensor) and in Einstein spaces, and, therefore, it can lead to conserved s-e currents. We shall see in Section 2 that, in fact, all the currents constructed from the Bel tensor (in vacuum or not) and a Killing vector satisfy properties analogous to those of Result 1.1, which will result in divergence-free currents in many general situations. Some specific examples of these currents are written down and analysed in Section 3 leading to physical consequences of some interest.

Nevertheless, in completely general situations, the Bel s-e currents will not be divergence-free; something which seems natural if the s-e concept is to have any physical meaning at all.

This is because if in the spacetime other fields than the gravitational one are present, one should expect in principle that the conserved current be formed by a combination of the sum of the gravitational Bel s-e current and the s-e current of the matter fields involved, in analogy with the energy-momentum currents, as explained before. This is supported also by the fact that the s-e currents of the matter fields are conserved in *flat* spacetime –that is, in the absence of gravitational field–, see for a proof [29, 28, 30]. As a matter of fact, the existence of such *mixed* divergence-free currents has been rigorously proved in [29, 28] for the case of a minimally coupled scalar field, that is, for the Einstein-Klein-Gordon situation, whenever there is a Killing vector. As explicitly shown in [29, 28], none of the two single currents, the scalar field s-e current nor the Bel one, are divergence-free separately in general. These results are summarised briefly in Section 4.

One of the purposes of this paper is to compute these divergence-free mixed currents in some particular situations, thereby providing explicit examples of the exchange of s-e between the scalar and gravitational fields, and the conservation of combined quantities. However, there arise several technical difficulties. First of all, the result for the Bel tensor which is analogous to Result 1.1 above, presented in Results 2.1 and 2.2 of Section 2, implies that (see Theorem 2.1): 1) if the Killing vector is hypersurface-orthogonal, then its corresponding Bel current is automatically divergence-free; and 2) if two Killing vectors form an Abelian group which acts orthogonally transitively on non-null surfaces, then all the corresponding Bel currents are divergence-free too. Thus, even though these results are physically interesting on their own, they prevent the existence of *mixed* currents in those cases. Hence, we have to resort to other more general cases in order to find such mixed currents. But then a second problem arises, namely, to find explicit spacetimes which are solution of the Einstein-Klein-Gordon equations under the needed more general circumstances: there are hardly any known solutions with a non-hypersurface-orthogonal Killing vector, and very few with a non-orthogonally transitive 2-parameter group.

The good news is, though, that there is at least an explicitly known family of solutions of the type we need. These solutions were found by Wils in [34] and we will use them in the second part of Section 4 to construct the sought *mixed* divergence-free s-e currents. This will provide explicit examples of the s-e interchange and of conserved currents which would have been very difficult to find, and even more to be singled out, if the concept of superenergy tensors had not been available.

2 General results on Bel currents

We will use the notation and terminology of [28] for the superenergy tensors. The Bel tensor, introduced by Bel more than 40 years ago [2, 3], is the basic superenergy tensor of the total gravitational field, that is to say, the s-e tensor $T_{\alpha\beta\lambda\mu}\left\{R_{[2],[2]}\right\}$ associated to the Riemann tensor $R_{\alpha\beta\lambda\mu}$. Its explicit expression reads [10, 8, 28]

$$\begin{aligned} B_{\alpha\beta\lambda\mu} \equiv R_{\alpha\rho\lambda\sigma}R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\alpha\rho\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \\ -\frac{1}{2}g_{\alpha\beta}R_{\rho\tau\lambda\sigma}R^{\rho\tau}{}_{\mu}{}^{\sigma} - \frac{1}{2}g_{\lambda\mu}R_{\alpha\rho\sigma\tau}R_{\beta}{}^{\rho\sigma\tau} + \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}R_{\rho\tau\sigma\nu}R^{\rho\tau\sigma\nu}, \end{aligned} \tag{2}$$

from where the following properties explicitly arise:

$$B_{\alpha\beta\lambda\mu} = B_{(\alpha\beta)(\lambda\mu)} = B_{\lambda\mu\alpha\beta}, \quad B^{\rho}{}_{\rho\lambda\mu} = 0. \tag{3}$$

Using the second Bianchi identity $\nabla_{[\nu} R_{\alpha\beta]\lambda\mu} = 0$ one obtains the following expression for the divergence of the Bel tensor:

$$\nabla_\alpha B^{\alpha\beta\lambda\mu} = R^{\beta}{}_{\rho}{}^{\lambda}{}_{\sigma} J^{\mu\sigma\rho} + R^{\beta}{}_{\rho}{}^{\mu}{}_{\sigma} J^{\lambda\sigma\rho} - \frac{1}{2} g^{\lambda\mu} R^{\beta}{}_{\rho\sigma\gamma} J^{\sigma\gamma\rho}, \tag{4}$$

where $J_{\lambda\mu\beta} = -J_{\mu\lambda\beta} \equiv \nabla_\lambda R_{\mu\beta} - \nabla_\mu R_{\lambda\beta}$ and $R_{\beta\mu}$ is the Ricci tensor. Note that because of (3) this is the only independent divergence of the Bel tensor. From (4) we obtain the fundamental result that B is divergence-free when $J_{\lambda\mu\beta} = 0$. This happens, for instance, in all Einstein spaces (where $R_{\mu\nu} = \Lambda g_{\mu\nu}$), so that in particular this implies that the Bel-Robinson tensor [1, 3, 4, 25], which is the s-e tensor of the Weyl conformal tensor and coincides with B in Ricci-flat spacetimes [8], is divergence-free. The divergence-free property of the Bel or Bel-Robinson tensors in these cases allows for the construction of divergence-free currents whenever there is a Killing vector.

These currents are built in a similar way to those formed with the energy-momentum tensor, as we show next. Following [28] one can define the *Bel current* with respect to any three Killing vector fields $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ (they do not need to be different!) as

$$j_\mu \left(\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3\right) \equiv B_{(\alpha\beta\lambda)\mu}\, \xi_1^\alpha \xi_2^\beta \xi_3^\lambda = B_{(\alpha\beta\lambda\mu)}\, \xi_1^\alpha \xi_2^\beta \xi_3^\lambda. \tag{5}$$

To avoid unnecessary writing we will omit the repetition of the Killing vectors: $\boldsymbol{j}\left(\vec{\xi}_1, \vec{\xi}_2\right) \equiv \boldsymbol{j}\left(\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_2\right)$ and $\boldsymbol{j}\left(\vec{\xi}\right) \equiv \boldsymbol{j}\left(\vec{\xi}, \vec{\xi}, \vec{\xi}\right)$. The divergence of these currents can be easily computed to give

$$\nabla_\mu j^\mu \left(\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3\right) = \nabla^\mu B_{(\alpha\beta\lambda\mu)}\, \xi_1^\alpha \xi_2^\beta \xi_3^\lambda + 3 B_{(\alpha\beta\lambda\mu)} \nabla^{(\mu} \xi_{(1}^{\alpha)} \xi_2^\beta \xi_{3)}^\lambda = \nabla^\mu B_{(\alpha\beta\lambda\mu)}\, \xi_1^\alpha \xi_2^\beta \xi_3^\lambda,$$

where we have used the fact that the $\vec{\xi}_a$ are Killing vectors: $\nabla^{(\alpha} \xi_a^{\beta)} = 0$ $(a = 1, 2, 3.)$ Therefore, the vanishing of the divergence of the Bel tensor implies the vanishing of that of $\vec{j}\left(\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3\right)$, defining, thereby, a conserved current.[3]

Nevertheless, the Bel currents are *not only* conserved when the Bel tensor is divergence-free. Following the general arguments in [9] showing the invertibility of orthogonal transitive Abelian G_2 groups (and, in particular, integrable 1-dim groups) acting on non-null orbits, since the Bel tensor is intrinsically defined, if the metric admits an invertible group then the Bel tensor must be invertible implying that its contraction with any odd number of Killing vectors provides a tensor tangent to the group orbits. Therefore, the s-e currents constructed from a hypersurface-orthogonal Killing vector or two non-null Killing vectors generating an Abelian orthogonally-transitive G_2 will necessarily be tangent to the orbits of the corresponding groups. This is stated as follows:

Result 2.1 *If $\vec{\xi}$ is a hypersurface-orthogonal Killing vector, then $\boldsymbol{j}\left(\vec{\xi}\right) \wedge \boldsymbol{\xi} = 0$.*

Result 2.2 *Let $\vec{\xi}$ and $\vec{\eta}$ be two independent commuting Killing vector fields spanning non-null surfaces. Then, if they act orthogonally transitively, their four associated Bel currents satisfy*

$$\boldsymbol{j}\left(\vec{\xi}\right) \wedge \boldsymbol{\xi} \wedge \boldsymbol{\eta} = \boldsymbol{j}\left(\vec{\eta}\right) \wedge \boldsymbol{\xi} \wedge \boldsymbol{\eta} = \boldsymbol{j}\left(\vec{\xi}, \vec{\eta}\right) \wedge \boldsymbol{\xi} \wedge \boldsymbol{\eta} = \boldsymbol{j}\left(\vec{\eta}, \vec{\xi}\right) \wedge \boldsymbol{\xi} \wedge \boldsymbol{\eta} = 0. \tag{6}$$

[3] Observe that *only* in strictly Ricci-flat spacetimes we can use conformal Killing vectors for $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ obtaining also local currents of type (5), as only in that case the Bel-Robinson tensor is divergence- *and* trace-free. Then, one can involve conformal Killing tensors—or even conformal Yano-Killing tensors as in [15]–.

These currents satisfy then properties completely analogous to that of the energy-momentum currents stated in Result 1.1, and therefore they will be divergence-free in quite general cases, as we will show below. For the sake of completeness, we provide the identities involving the s-e currents and the geometrical properties of 1-dim and 2-dim groups of isometries, from which Results 2.1 and 2.2 trivially follow.

Identity 2.1 *Let $\vec{\xi}$ be a Killing vector field in $(\mathcal{V}, g)$. Then we have*

$$\xi^\alpha \xi^\beta \xi^\lambda B_{\alpha\beta\lambda[\mu}\xi_{\gamma]} = \frac{3}{2}\left\{\xi_\lambda \xi_\beta R^{\lambda\sigma\beta\tau}\nabla_\tau\left(\xi_{[\mu}\xi_{\gamma;\sigma]}\right) + \xi^\beta R^{\rho\tau\sigma}{}_{[\mu}\xi_{\gamma]}\nabla_\sigma\left(\xi_{[\rho}\xi_{\beta;\tau]}\right)\right\}. \tag{7}$$

Proof. Since $\vec{\xi}$ is a Killing vector we have [12, 22]

$$\nabla_\alpha\nabla_\beta\xi_\lambda = \xi_\rho R^\rho{}_{\alpha\beta\lambda}, \tag{8}$$

so that direct contraction of the Bel tensor (2) with $\vec{\xi}$ yields, on using (8),

$$\begin{aligned}\xi^\alpha\xi^\beta\xi^\lambda B_{\alpha\beta\lambda\mu} =& 2\xi^\lambda\nabla_\rho\nabla_\sigma\xi_\lambda\nabla^\rho\nabla^\sigma\xi_\mu + \frac{1}{2}\xi^\lambda\xi_\lambda\nabla_\sigma\nabla_\rho\xi_\tau R^{\rho\tau\sigma}{}_\mu - \\ &-\frac{1}{2}\xi_\mu\nabla_\rho\nabla_\sigma\xi_\lambda\nabla^\rho\nabla^\sigma\xi^\lambda + \frac{1}{8}\xi^\lambda\xi_\lambda\xi_\mu R^{\rho\sigma\tau\nu}R_{\rho\sigma\tau\nu}.\end{aligned}$$

The exterior product of this one-form with $\boldsymbol{\xi}$ gives then

$$\xi^\alpha\xi^\beta\xi^\lambda B_{\alpha\beta\lambda[\mu}\xi_{\gamma]} = 2\xi_\lambda\nabla^\rho\nabla^\sigma\xi^\lambda\nabla_\rho\nabla_\sigma\xi_{[\mu}\xi_{\gamma]} + \frac{1}{2}\xi^\lambda\xi_\lambda\nabla_\sigma\nabla_\rho\xi_\tau R^{\rho\tau\sigma}{}_{[\mu}\xi_{\gamma]}. \tag{9}$$

Expanding the covariant derivative of the three-form $\boldsymbol{\xi}\wedge d\boldsymbol{\xi}$ we get

$$3\ \nabla_\rho\left(\xi_{[\gamma}\xi_{\sigma;\mu]}\right) = -3\ \nabla_\gamma\xi_{[\rho}\nabla_\mu\xi_{\sigma]} + \xi_\sigma\nabla_\rho\nabla_\gamma\xi_\mu - 2\ \nabla_\rho\nabla_\sigma\xi_{[\mu}\xi_{\gamma]}. \tag{10}$$

and taking into account that $\xi_\lambda\nabla^\rho\nabla^\sigma\xi^\lambda$ is symmetric due to (8), its contraction with (10) provides

$$3\ \xi_\lambda\nabla^\rho\nabla^\sigma\xi^\lambda\nabla_\rho\left(\xi_{[\gamma}\xi_{\sigma;\mu]}\right) = -2\ \xi_\lambda\nabla^\rho\nabla^\sigma\xi^\lambda\nabla_\rho\nabla_\sigma\xi_{[\mu}\xi_{\gamma]}. \tag{11}$$

Let $M^{\{\Omega\}\rho\mu\sigma}$ be any tensor with an arbitrary number of indices (denoted by $\{\Omega\}$) plus three indices such that $M^{\{\Omega\}[\rho\mu\sigma]} = M^{\{\Omega\}\rho(\mu\sigma)} = 0$. After relabelling the indices, the contraction of (10) with $M^{\{\Omega\}\rho\mu\sigma}$ and ξ^γ gives

$$3\ \xi^\beta M^{\{\Omega\}\sigma\rho\tau}\nabla_\sigma\left(\xi_{[\beta}\xi_{\tau;\rho]}\right) = \xi^\lambda\xi_\lambda\nabla_\sigma\nabla_\rho\nabla_\tau M^{\{\Omega\}\sigma\rho\tau} + 2\ \xi^\beta\nabla_\sigma\nabla_\tau\xi_\beta\xi_\rho M^{\{\Omega\}\sigma\rho\tau}. \tag{12}$$

Replacing now $M^{\{\Omega\}\sigma\rho\tau}$ by $R^{\rho\tau\sigma}{}_{[\mu}\xi_{\gamma]}$ and using (8) and (11) we find

$$3\ \xi^\beta R^{\rho\tau\sigma}{}_{[\mu}\xi_{\gamma]}\nabla_\sigma\left(\xi_{[\beta}\xi_{\tau;\rho]}\right) = \xi^\lambda\xi_\lambda\nabla_\sigma\nabla_\rho\xi_\tau R^{\rho\tau\sigma}{}_{[\mu}\xi_{\gamma]} - 3\ \xi_\lambda\nabla^\rho\nabla^\sigma\xi^\lambda\nabla_\rho\left(\xi_{[\gamma}\xi_{\sigma;\mu]}\right). \tag{13}$$

The final step is just to compute the linear combination (9)+(11)$-\frac{1}{2}$ (13) to obtain

$$\xi^\alpha\xi^\beta\xi^\lambda B_{\alpha\beta\lambda[\mu}\xi_{\gamma]} = \frac{3}{2}\left\{-\xi_\lambda\nabla^\rho\nabla^\sigma\xi^\lambda\nabla_\rho\left(\xi_{[\gamma}\xi_{\sigma;\mu]}\right) + \xi^\beta R^{\rho\tau\sigma}{}_{[\mu}\xi_{\gamma]}\nabla_\sigma\left(\xi_{[\beta}\xi_{\tau;\rho]}\right)\right\},$$

which, after using (8) and some index manipulation, gives the desired identity (7). ■

Note that $\xi^\alpha\xi^\beta\xi^\lambda B_{\alpha\beta\lambda\mu} = \xi^\alpha\xi^\beta\xi^\lambda B_{(\alpha\beta\lambda)\mu} = \xi^\alpha\xi^\beta\xi^\lambda B_{(\alpha\beta\lambda\mu)}$ due to the symmetries in (3). If $\vec{\xi}$ is hypersurface-orthogonal, i.e. $\xi_{[\rho}\xi_{\beta;\mu]} = 0$, Result 2.1 explicitly follows at once as a corollary of formula (7).

Identity 2.2 *Let $\vec{\xi}$ and $\vec{\eta}$ be two Killing vector fields in $(\mathcal{V}, g)$ generating a two dimensional group of isometries acting on non-null surfaces. Taking into account (33), (40), (36)-(39) in the Appendix, the four associated currents can be expressed as follows*

$$
\begin{aligned}
&-3W^2\, j_{[\alpha}\left(\vec{\xi}_A, \vec{\xi}_B\right) w_{\mu\nu]} = \\
&2\frac{3!}{W}\left\{\left(\xi_A^{(\lambda}\xi_{A\gamma}w^{\tau)\gamma}\xi_B^{\epsilon} + 2\xi_B^{(\lambda}\xi_{A\gamma}w^{\tau)\gamma}\xi_A^{\epsilon}\right)(\eta_\lambda\tilde{\Omega}_{1\rho} - \xi_\lambda\tilde{\Omega}_{2\rho})R_\tau{}^\rho{}_\epsilon{}^\sigma * w_{\sigma[\alpha}w_{\mu\nu]} + \right. \\
&\left. + \left(\xi_A^{(\lambda}\xi_{A\gamma}w^{\tau)\gamma}\xi_{B\beta}w^{\epsilon\beta} + 2\xi_A^{(\lambda}\xi_{B\gamma}w^{\tau)\gamma}\xi_{A\beta}w^{\epsilon\beta}\right)(\eta_\lambda\tilde{\Sigma}_{1\rho} - \xi_\lambda\tilde{\Sigma}_{2\rho})R_\tau{}^\rho{}_{\epsilon[\alpha}w_{\mu\nu]}\right\} + \\
&+\frac{3}{2}\left[\left(2\,(\vec{\xi}_A\cdot\vec{\xi}_A)\tilde{\Sigma}_{B\tau} + 4(\vec{\xi}_A\cdot\vec{\xi}_B)\tilde{\Sigma}_{A\tau}\right)w^{\lambda\sigma} + \right. \\
&\left. + \left((\vec{\xi}_A\cdot\vec{\xi}_A)\tilde{\Omega}_{B\tau} + 2(\vec{\xi}_A\cdot\vec{\xi}_B)\tilde{\Omega}_{A\tau}\right) * w^{\lambda\sigma}\right] R_{\lambda\sigma}{}^\tau{}_{[\alpha}w_{\mu\nu]} + \\
&-\frac{1}{8}\left(2w^{\lambda\sigma} + *w^{\lambda\sigma}\right)w^{\beta\gamma}R_{\beta\gamma\lambda\sigma}\left((\vec{\xi}_A\cdot\vec{\xi}_A)\xi_B^\tau + 2(\vec{\xi}_A\cdot\vec{\xi}_B)\xi_A^\tau\right)(\eta_\tau\Sigma_{1[\alpha}w_{\mu\nu]} - \xi_\tau\Sigma_{2[\alpha}w_{\mu\nu]}),
\end{aligned}
\tag{14}
$$

where the indices A, B, C take the values $1, 2$ to denote $\vec{\xi}$ and $\vec{\eta}$ respectively, and $*$ denotes the usual Hodge dual.

Proof. Not to overwhelm the text, we prefer to present the proof in an Appendix. ■

Clearly, if the G_2 isometry group is Abelian and acts orthogonally transitively, then $\boldsymbol{\Sigma}_A \wedge \boldsymbol{w} = \boldsymbol{\Omega}_A \wedge \boldsymbol{w} = 0$ and $\tilde{\boldsymbol{\Sigma}}_A = \tilde{\boldsymbol{\Omega}}_A = 0$ for $A = 1, 2$, as follows from (36)-(39), and Result 2.2 is recovered.

Let us introduce the notation $\vec{j}_\Upsilon$, $(\Upsilon = 1, 2, 3, 4)$, for the four Bel currents appearing in (6). Then, Result 2.2 implies that

$$
\vec{j}_\Upsilon = a_\Upsilon(x)\vec{\xi} + b_\Upsilon(x)\vec{\eta}. \tag{15}
$$

The functions a_Υ and b_Υ are not arbitrary. Note that, given that $\vec{\xi}$ and $\vec{\eta}$ are Killing vectors, then $\pounds_{\vec{\xi}} R^\alpha{}_{\beta\lambda\mu} = \pounds_{\vec{\eta}} R^\alpha{}_{\beta\lambda\mu} = 0$, and, therefore,

$$
\pounds_{\vec{\xi}} B_{\alpha\beta\lambda\mu} = \pounds_{\vec{\eta}} B_{\alpha\beta\lambda\mu} = 0. \tag{16}
$$

Hence, if we compute the Lie derivative with respect to both Killing vector fields of equation (15), by using (16) and the definition (5) of Bel currents we deduce

$$
0 = \pounds_{\vec{\xi}}\vec{j}_\Upsilon = \left(\pounds_{\vec{\xi}}\, a_\Upsilon\right)\vec{\xi} + \left(\pounds_{\vec{\xi}}\, b_\Upsilon\right)\vec{\eta}
$$

and a similar relation replacing $\pounds_{\vec{\xi}}$ by $\pounds_{\vec{\eta}}$. In consequence, the functions a_Υ and b_Υ are restricted by

$$
\pounds_{\vec{\xi}}\, a_\Upsilon = \pounds_{\vec{\xi}}\, b_\Upsilon = \pounds_{\vec{\eta}}\, a_\Upsilon = \pounds_{\vec{\eta}}\, b_\Upsilon = 0.
$$

Taking now the divergence of equation (15) and using this result we arrive at

$$
\nabla_\rho j^\rho_\Upsilon = 0 \qquad \Upsilon = 1, 2, 3, 4\ .
$$

The case with only one hypersurface-orthogonal Killing vector can be treated as a special case of equation (15) with $\Upsilon = 1$ and $b_1 = 0$. All these results can be summarised as follows:

Theorem 2.1 *1. If $\vec{\xi}$ is a hypersurface-orthogonal Killing vector then its corresponding Bel current $\vec{j}\left(\vec{\xi}\right)$ is proportional to $\vec{\xi}$ and divergence-free.*

2. *If $\vec{\xi}$ and $\vec{\eta}$ are two commuting Killing vectors acting orthogonally transitively on non-null surfaces, then their four corresponding Bel currents $\vec{j}_\Upsilon$ lie in the 2-planes generated by $\{\vec{\xi},\vec{\eta}\}$ and are divergence-free.*

These results are very interesting, and reinforce, once more, the various analogies between the energy-momentum tensors and some of the superenergy tensors. Nevertheless, as remarked in the Introduction, they forbid the existence of *mixed* superenergy currents if they are to be constructed with Killing fields of the type appearing in the results. Unfortunately, most of the known Einstein-Klein-Gordon exact solutions contain such types of Killing vector fields. In order to get a flavour of the possible interpretation of the s-e currents, we are going to present some simple cases of pure divergence-free currents in the next section 3, leaving the more difficult mixed case for section 4, in which an explicit solution of the field equations without the above mentioned properties for the Killing vector fields will be used.

3 Bel and Bel-Robinson currents in Reissner-Nordström and Schwarzschild spacetimes

Let us consider the Reissner-Nordström spacetime, whose line-element, in standard spherical coordinates $\{t, r, \theta, \varphi\}$, reads [13, 22, 16]

$$ds^2 = -\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)dt^2+\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2, \tag{17}$$

where m and q are arbitrary constants representing the total mass and electric charge of the particle creating the gravitational field, and $d\Omega^2 = d\theta^2+\sin^2\theta\, d\varphi^2$ is the canonical line-element in the unit 2-sphere. We only consider the exterior asymptotically flat region defined by $1-\frac{2m}{r}+\frac{q^2}{r^2}>0$ which restricts the range of the coordinate r to $r>0$ if $m^2<q^2$, or to $r>r_+\equiv m+\sqrt{m^2-q^2}$ if $m^2\geq q^2$. The metric (17) is the general spherically symmetric solution of the Einstein-Maxwell equations. The electromagnetic field is given by $\boldsymbol{F}=(q/r^2)\; dt\wedge dr$, and its energy-momentum tensor expressed (in natural units $8\pi G=c=1$ where c is the speed of light in vacuum and G is the gravitational constant) in the coordinate basis is

$$T^\mu{}_\nu = \frac{q^2}{r^4}\times \mathrm{diag}\{-1,-1,1,1\}\,. \tag{18}$$

The particular case with $q=0$ is the vacuum Schwarzschild solution (then $r>2m$).

In the chosen range of coordinates, the metric is static ($\vec{\xi}=\partial/\partial t$ is a hypersurface-orthogonal timelike Killing vector) and spherically symmetric. We enumerate the four Killing vectors $\vec{\xi}_L$, $(L=1,2,3,4)$ as

$$\vec{\xi}_1=\frac{\partial}{\partial t},\quad \vec{\xi}_2=\sin\varphi\frac{\partial}{\partial\theta}+\cos\varphi\cot\theta\frac{\partial}{\partial\varphi},\quad \vec{\xi}_3=\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi},\quad \vec{\xi}_4=\frac{\partial}{\partial\varphi}. \tag{19}$$

We want to study the Bel (and Bel-Robinson for the $q=0$-case) currents constructed with the Killing vectors (19) for these spacetimes, together with the currents arising from the energy-momentum tensor. These will lead to some conserved quantities. Of course, one expects that all the conserved quantities will be just functions of the constants m and q, but our aim is to

ascertain which particular combinations of m and q arise at the energy and superenergy levels. To find them we compute explicitly all the Bel currents:

$$\boldsymbol{j}(\vec{\xi}_1) = -\frac{1}{r^6}\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)\left[6\left(m-\frac{q^2}{r}\right)^2+\frac{q^4}{r^2}\right]\boldsymbol{\xi}_1, \tag{20}$$

$$\begin{aligned}
\boldsymbol{j}\left(\vec{\xi}_a,\vec{\xi}_1\right) &= \frac{1}{3r^6}\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)\left[12\left(m-\frac{q^2}{r}\right)^2+\frac{q^4}{r^2}\right]\boldsymbol{\xi}_a,\\
\boldsymbol{j}\left(\vec{\xi}_a,\vec{\xi}_b,\vec{\xi}_1\right) &= -\frac{1}{3r^6}\left(\vec{\xi}_a\cdot\vec{\xi}_b\right)\left[12\left(m-\frac{q^2}{r}\right)^2+\frac{q^4}{r^2}\right]\boldsymbol{\xi}_1, \qquad (21)\\
\boldsymbol{j}\left(\vec{\xi}_a,\vec{\xi}_b,\vec{\xi}_c\right) &= \frac{1}{3r^6}\left[6\left(m-\frac{q^2}{r}\right)^2+\frac{q^4}{r^2}\right]\left\{\left(\vec{\xi}_a\cdot\vec{\xi}_b\right)\boldsymbol{\xi}_c+\left(\vec{\xi}_b\cdot\vec{\xi}_c\right)\boldsymbol{\xi}_a+\left(\vec{\xi}_c\cdot\vec{\xi}_a\right)\boldsymbol{\xi}_b\right\},
\end{aligned}$$

where $a,b,c\in\{2,3,4\}$. Similarly, the energy-momentum currents defined in (1) can be computed on this metric and they read:

$$\begin{aligned}
\boldsymbol{\mathcal{J}}\left(\vec{\xi}_1\right) &= -\frac{q^2}{r^4}\boldsymbol{\xi}_1 \qquad (22)\\
\boldsymbol{\mathcal{J}}\left(\vec{\xi}_a\right) &= \frac{q^2}{r^4}\boldsymbol{\xi}_a.
\end{aligned}$$

It can be easily checked that all the above currents are divergence-free, and therefore they lead to a set of conserved quantities via Gauss' theorem applied to appropriate compact 4-volumes of the spacetime. For illustration purposes, we are going to compute now these conserved quantities for the simple but relevant case in which the compact region is taken to be bounded by two $t=$const. hypersurfaces, say $t=t_1$ and $t=t_2$, $t_2>t_1$, and two $r=$ const. hypersurfaces, say $r=r_1$ and $r=r_2$, $r_2>r_1$. We call this region $\mathcal{K}$. Applying Gauss' theorem we immediately get

$$\int_{\partial\mathcal{K}} A^\mu n_\mu \,\mathrm{d}^3\sigma = 0,$$

where $\vec{A}$ represents any of the $\vec{j}$ or $\vec{\mathcal{J}}$ computed above, $\partial\mathcal{K}$ denotes the boundary of $\mathcal{K}$, $\boldsymbol{n}$ is the outward unit normal to $\mathcal{K}$ and $\mathrm{d}^3\sigma$ is the canonical volume 3-element on $\partial\mathcal{K}$. Clearly, $\partial\mathcal{K}=\{t=t_1,r_1<r<r_2\}\cup\{t=t_2,r_1<r<r_2\}\cup\{r=r_1,t_1<t<t_2\}\cup\{r=r_2,t_1<t<t_2\}$, so that the corresponding unit normals are proportional to dt for the first two regions, and to dr for the remaining two. From the explicit expressions of the currents, we know that none of them has a non-zero component along dr, and therefore the integrals on $\{r=r_1\}\cup\{r=r_2\}$ vanish. Thus, only the integrals on $\{t=t_1\}$ and $\{t=t_2\}$ remain, and thus they must be equal. In other words, the integrals

$$\int_{t=\text{const.}} A^\mu n_\mu \,\mathrm{d}^3\sigma, \tag{23}$$

taken over any portion of a $t=$const. hypersurface bounded by the values r_1 and r_2, are constant in the sense that they are independent of the particular $t=$const. hypersurface. It is quite remarkable that the integrand in (23) for the case of the electromagnetic current (22) is *not* the standard expression for the electromagnetic energy density, taken usually as $T^{\mu\nu}n_\mu n_\nu$. However, (23) leads to simpler results and to expressions which are apparently correct (at least in a naive

comparison with the analogous ones in the classical theory) such as (25) and (26) found later. Thus, one wonders if, in a stationary frame of reference defined by $\vec{\xi}$, the correct expression for the electromagnetic energy density taken over an extended region of the spacetime —and not only at a point— should be in fact $T^{\mu\nu}\xi_\mu n_\nu$.

Coming back to (23), and by noting that $\boldsymbol{n} = -\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^{1/2} dt$ and that $\mathrm{d}^3\sigma = \left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^{-1/2} r^2 \sin\theta dr d\theta d\varphi$, those integrals reduce to

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_{r_1}^{r_2} (-A^t) r^2 dr, \tag{24}$$

where A^t denotes the t-component of $\vec{A}$. Observe that, once again, among the explicit expressions for $\vec{j}$ and $\vec{\mathcal{J}}$, only those involving $\vec{\xi}_1$ have a non-zero t-component, and therefore they are the only ones which may give *non-trivial* conserved quantities: all other currents provide constants which are simply identically zero. Actually, not even all of the currents involving $\vec{\xi}_1$ provide a non-zero constant, because the angular integrals can also vanish. In summary, it is easy to check that the only non-trivial constants arise from the currents (22), (20) and the three with $a = b$ in (21), these last three being, in fact, equal. Denoting them by $\mathcal{Q}_1$, Q_1 and Q_2 respectively, they clearly depend on the values of r_1 and r_2. Their explicit expressions are:

$$\mathcal{Q}_1 = 4\pi q^2 \left(\frac{1}{r_1} - \frac{1}{r_2}\right), \tag{25}$$

$$Q_1 = \frac{4\pi}{15\, r^7}\left(45m^3r^3 - 90m^2q^2r^2 - 30m^2r^4 + 65mq^4r + 45mq^2r^3 - 21q^4r^2 - 15q^6\right)\Big|_{r_1}^{r_2},$$

$$Q_2 = -\frac{8\pi}{27r^3}\left(36m^2r^2 - 36mq^2r + 13q^4\right)\Big|_{r_1}^{r_2}.$$

It is also remarkable that these three expressions are strictly positive. This is a general result—independent of the spacetime if it is stationary— for the currents of type $\mathcal{Q}_1$ and Q_1, and follows from the dominant energy condition satisfied by $T^{\mu\nu}$ in the first case, and from the analogous dominant property satisfied by the Bel tensor [24, 6, 7, 28] in the second, as the corresponding integrands are positive. This was noted for the Bel-Robinson case for instance in [11] (see also the recent generalisation in [15]). Nevertheless, there is no similar reasoning for Q_2 and its positivity can only be inferred from the explicit expression of the local current (21).

If we want to obtain constants associated to the spacetime, we can also take the limit cases in which the above expressions are computed over a whole slice t =const. by taking $r_2 \to \infty$ and the minimum possible value for r_1. There appear then two different situations depending on whether $m^2 \geq q^2$ or not. If $m^2 < q^2$, then the minimum value for r_1 is $r_1 = 0$, and clearly all the above constants diverge. This is natural and analogous to what happens in flat spacetime if we integrate the energy over the whole space. However, for the case with $m^2 \geq q^2$, the existence of the event horizon at $r_+ = m + \sqrt{m^2 - q^2}$ provides a finite minimum value for r_1, and this leads to finite conserved constants. A simple computation leads to

$$\begin{aligned}
\mathcal{Q}_1 &= \frac{4\pi q^2}{r_+}, \\
Q_1 &= \frac{4\pi}{15\, r_+}\left(6 - 14\frac{m}{r_+} + 14\frac{m^2}{r_+^2} - 5\frac{m^3}{r_+^3}\right),
\end{aligned}$$

$$Q_2 \quad = \quad \frac{8\,\pi}{27} r_+ \left(13 - 16 \frac{m}{r_+} + 16 \frac{m^2}{r_+^2} \right).$$

For the particular case of the Schwarzschild spacetime ($q = 0$), they become simply

$$\mathcal{Q}_1 = 0, \qquad Q_1 = \frac{\pi}{4m}, \qquad Q_2 = \frac{16\,\pi\,m}{3}.$$

An obvious question arises: are these *particular* combinations of m and q special in any sense? And if yes, why? Clearly, the meaning of $\mathcal{Q}_1$ is the total electrostatic energy of the Reissner-Nordström black hole with respect to the static observer. However, in order to check this simple statement one has to put back all the physical constants which were implicitly taken to be unity. Denoting by M and Q the total mass and charge in the correct units, respectively, we have $m = GM/c^2$ and $q^2 = GQ^2/c^4$. Now, the energy-momentum tensor (18) must be multiplied by $c^4/(8\pi G)$, and therefore we finally get for the physical $\mathcal{Q}_1$:

$$\mathcal{Q}_1 = \frac{1}{2} \frac{Q^2}{r_+}. \tag{26}$$

This is a satisfactory result which may provide a physical interpretation for $\mathcal{Q}_1$, as this formula is reminiscent of that for the energy of the electric field in classical physics.

Now, the same type of reasoning is needed for the superenergy quantities Q_1 and Q_2. To that end, we need to know which type of physical dimensions are carried by the superenergy tensors. This can be deduced from several independent works [14, 5, 30, 17], and as is explained in [28] (p. 2820) the correct physical units for the *physical* Bel tensor seem to be energy density per unit surface, i.e. $ML^{-3}T^{-2}$. This means that we have to multiply the Bel tensor by $\kappa c^4/G$ to get the correct physical quantities, where κ is a pure number to be chosen. Doing that we can obtain the following expression for the physical Q_1:

$$Q_1 = \kappa \frac{4\pi}{r_+^2} \left(\frac{1}{4} Mc^2 + \beta \frac{Q^2}{r_+} \right) = \kappa \frac{4\pi}{r_+^2} \left(\frac{1}{4} Mc^2 + 2\beta \mathcal{Q}_1 \right) \tag{27}$$

where $\beta = -\frac{1}{2} \left(\frac{1}{3} \frac{m^2}{r_+^2} - \frac{23}{30} \frac{m}{r_+} + \frac{4}{5} \right)$ is a dimensionless quantity. Of course, since (27) is written in terms of three non-independent quantities, M, Q and r_+, it is not unique. It only provides a possible good choice. For the particular case of Schwarzschild's solution ($q = 0 \Rightarrow r_+ = 2m$) the previous quantity reduces simply to

$$Q_1 = \kappa \frac{\pi}{4m^2} Mc^2 .$$

We note in passing that if we chose $\kappa = (2\pi)^{-2}$ we could rewrite this as

$$Q_1 = \frac{Mc^2}{\mathcal{A}} \tag{28}$$

where $\mathcal{A}$ is the area of the horizon. Nevertheless, such an amusing or inspiring interpretation for Q_1 is not possible in the general case (27) due to the factor β.

A possible way to try and obtain alternative, more complete, expressions of type (27) is by using the existence of independent superenergy of pure electromagnetic origin, see [28]. In order

to check this possibility, we can use any of the s-e tensors for the electromagnetic field (the most general one depends on six arbitrary constants, see [27, 28]) because the corresponding current constructed with $\vec{\xi}_1$ is always the same, as can be easily checked. Thus, we can use the basic s-e tensor $T_{\alpha\beta\lambda\mu}\{\nabla_{[1]}F_{[2]}\}$ for ∇F, given by [27, 28]

$$E_{\alpha\beta\lambda\mu} \equiv \nabla_\alpha F_{\lambda\rho}\nabla_\beta F_\mu{}^\rho + \nabla_\alpha F_{\mu\rho}\nabla_\beta F_\lambda{}^\rho - g_{\alpha\beta}\nabla_\sigma F_{\lambda\rho}\nabla^\sigma F_\mu{}^\rho - \\ -\frac{1}{2}g_{\lambda\mu}\nabla_\alpha F_{\sigma\rho}\nabla_\beta F^{\sigma\rho} + \frac{1}{4}g_{\alpha\beta}g_{\lambda\mu}\nabla_\tau F_{\sigma\rho}\nabla^\tau F^{\sigma\rho}\,. \tag{29}$$

Its symmetry properties are

$$E_{\alpha\beta\lambda\mu} = E_{(\alpha\beta)(\lambda\mu)}$$

but it is not symmetric in the exchange of $\alpha\beta$ with $\lambda\mu$ (a tensor with that symmetry was found many years ago by Chevreton [10, 30]; Chevreton's tensor is simply proportional to $E_{\alpha\beta\lambda\mu} + E_{\lambda\mu\alpha\beta}$. In what follows, one can use either of these tensors or, for that matter, its completely symmetric part). The current associated to (29), and analogous to (20), can be easily computed to produce

$$E^\alpha{}_{\beta\lambda\mu}\xi_1^\beta\xi_1^\lambda\xi_1^\mu = -\frac{3q^2}{r^6}\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^2\xi_1^\alpha, \tag{30}$$

so that it is clearly divergence-free too. Thus, by a similar procedure as above it gives rise to a conserved quantity when integrated over any t =constant spatial section. Its value is

$$Q_1^{(F)} = \frac{4\pi\, q^2}{35\, r_+^3}\left(4\frac{m^2}{r_+^2} - 11\frac{m}{r_+} + 8\right) = \frac{4\pi}{35\, r_+}\left(8\frac{m^3}{r_+^3} - 26\frac{m^2}{r_+^2} + 27\frac{m}{r_+} - 8\right). \tag{31}$$

Therefore, by forming linear combinations with positive coefficients c_1 and c_2 of the respective s-e tensors we can get a conserved quantity given by $c_1Q_1 + c_2Q_1^{(F)}$ which, when the physical units have been restored, becomes

$$c_1Q_1 + c_2Q_1^{(F)} = \kappa\frac{4\pi}{r_+^2}\left(A\,Mc^2 + B\frac{Q^2}{r_+}\right)$$

with

$$A = a + (b-2a)\frac{m}{r_+} + (c_1 - 2b)\frac{m^2}{r_+^2},$$

$$B = \frac{8}{35}c_2 - \frac{2}{5}c_1 + (a + \frac{2}{15}c_1 - \frac{11}{35}c_2)\frac{m}{r_+} + (b - \frac{2}{3}c_1 + \frac{4}{35}c_2)\frac{m^2}{r_+^2}.$$

Here a and b are spureous constants (due to the fact that $1-2m/r_+ + q^2/r_+^2 = 0$), so that these expressions can be simplified on using this freedom. One can try to restrict the values of the constants by using other kind of arguments. For instance, there are independent results that prove the conservation of the simply added superenergy along shock-wave discontinuities for Einstein-Maxwell spacetimes [20] (see also subsection 7.3 in [28] and references therein), hence it seems reasonable to put in any case $c_1 = c_2$. Setting then $c_1 = c_2 = 1$ and, as an example,

choosing a and b such that A does not depend on m/r_+ so that the case $Q = 0$ is explicitly recovered, we get

$$Q_1 + Q_1^{(F)} = \kappa \frac{4\pi}{r_+^2} \left(\frac{1}{4} Mc^2 + B \frac{Q^2}{r_+} \right),$$

where now $B = -\frac{1}{35} \left(\frac{11}{6} \frac{m^2}{r_+^2} - \frac{29}{12} \frac{m}{r_+} + 6 \right)$. Other choices for a and b lead to similar formulas. Thus, the interpretation of Q_1 and, in general, of its combinations with $Q_1^{(F)}$, remains somewhat obscure.

For the sake of completeness, we include now the physical Q_2 (note that the corresponding $Q_2^{(F)}$ vanishes, so that in this case the linear combinations of the superenergy tensors always lead to a quantity proportional to Q_2):

$$Q_2 = 8\pi\kappa \left(\delta Mc^2 + \gamma \frac{Q^2}{r_+} \right) = 8\pi\kappa \left(\delta Mc^2 + 2\gamma \mathcal{Q}_1 \right),$$

where now $\delta = 2(8/27 - k)m/r_+ + k + 10/27$ and $\gamma = k\, m/r_+ - 13/27$ and k is a disposable constant similar to the previous a and b. Choosing again δ not to depend on m/r_+, one gets

$$Q_2 = \kappa \frac{16\pi}{3} \left[Mc^2 + \frac{1}{9} \left(8 \frac{m}{r_+} - 13 \right) \mathcal{Q}_1 \right].$$

The units of this quantity are those of energy, that is ML^2T^{-2}, but given its origin and construction one has doubts about whether considering it as such or as a kind of "supermomentum" per unit surface: $(ML^2T^{-2})L^2/L^2$. In the Schwarzschild case, $q = 0$, one gets then

$$Q_2 = \kappa \frac{16\pi}{3} Mc^2.$$

4 Conserved mixed currents

As discussed in the Introduction, and remarked at the end of the previous section, if there are non-gravitational fields present in the spacetime then they may also contribute to the total superenergy tensor. One important question is whether or not the total s-e current thus constructed may be divergence-free. Of course, this depends on the field equations for the particular matter fields involved. In [29, 28], it was proved in full generality that these divergence-free *mixed* s-e currents can always be constructed for the simple case of a minimally coupled scalar field (whenever there is a Killing vector in the spacetime). We now summarize here the main results in [29, 28] concerning the massless case, which is the one relevant for our present purposes.

Let ϕ be a massless scalar field. The s-e tensor for the scalar field is the basic s-e tensor $T_{\alpha\beta\lambda\mu}\{\nabla_{[1]}\nabla_{[1]}\phi\}$ for $\nabla\nabla\phi$ and can be easily constructed following the general definition of [28]. The result is (see [27, 28, 29, 30]):

$$S_{\alpha\beta\lambda\mu} \equiv \nabla_\alpha \nabla_\lambda \phi \nabla_\mu \nabla_\beta \phi + \nabla_\alpha \nabla_\mu \phi \nabla_\beta \nabla_\lambda \phi - \\ - g_{\alpha\beta} \nabla_\lambda \nabla^\rho \phi \nabla_\mu \nabla_\rho \phi - g_{\lambda\mu} \nabla_\alpha \nabla^\rho \phi \nabla_\beta \nabla_\rho \phi + \frac{1}{2} g_{\alpha\beta} g_{\lambda\mu} \nabla_\sigma \nabla_\rho \phi \nabla^\sigma \nabla^\rho \phi \qquad (32)$$

from where we immediately deduce

$$S_{\alpha\beta\lambda\mu} = S_{(\alpha\beta)(\lambda\mu)} = S_{\lambda\mu\alpha\beta}.$$

One can straightforwardly compute the divergence of the scalar field s-e tensor

$$\nabla_\alpha S^\alpha{}_{\beta\lambda\mu} = 2\nabla_\beta\nabla_{(\lambda}\phi R_{\mu)\rho}\nabla^\rho\phi - g_{\lambda\mu}R^{\sigma\rho}\nabla_\beta\nabla_\rho\phi\nabla_\sigma\phi - \nabla_\sigma\phi(2\nabla^\rho\nabla_{(\lambda}\phi R^\sigma{}_{\mu)\rho\beta} + g_{\lambda\mu}R^\sigma{}_{\rho\beta\tau}\nabla^\rho\nabla^\tau\phi)$$

so that we realize that the s-e tensor (32) is divergence-free in flat spacetime, that is to say, in the absence of gravatational field. This allows to construct conserved currents for the scalar field in flat spacetime. They are built in a similar way to those formed with the energy-momentum tensor or the Bel currents (5). Following [29, 28, 30], one can define the *scalar-field s-e current* with respect to any three Killing vector fields $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ as

$$\daleth_\mu\left(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3\right) \equiv S_{(\alpha\beta\lambda)\mu}\,\xi_1^\alpha\xi_2^\beta\xi_3^\lambda = S_{(\alpha\beta\lambda\mu)}\,\xi_1^\alpha\xi_2^\beta\xi_3^\lambda.$$

Nevertheless, these currents are not divergence-free in curved spacetimes. In other words, they are not divergence-free if one takes into account the gravitational field created by the scalar field. This was to be expected: if the s-e concept is to have any physical meaning at all, then the *total* s-e currents, involving both the gravitational and the scalar fields, are the ones to be conserved. And this was actually proved in [29, 28]: if the Einstein-Klein-Gordon field equations are satisfied, that is

$$R_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi \qquad \Longrightarrow \qquad \nabla_\mu\nabla^\mu\phi = 0$$

then the sum of the Bel current and the scalar-field s-e current, which will be written as

$$\vec{J}(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3) \equiv \vec{j}(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3) + \vec{\daleth}(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3)$$

is a divergence-free vector field:

$$\nabla_\alpha J^\alpha(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3) = 0.$$

Observe that $\vec{J}$ is explicitly defined by

$$J_\mu\left(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3\right) \equiv \left(B_{(\alpha\beta\lambda)\mu} + S_{(\alpha\beta\lambda)\mu}\right)\xi_1^\alpha\xi_2^\beta\xi_3^\lambda\,.$$

These are mixed conserved currents, and they lead to the conservation of *mixed* quantities, containing both gravitational and scalar fields contributions. Moreover, note that, in general, none of the two single currents $\vec{\daleth}$ nor $\vec{j}$ are divergence-free separately.

We now want to produce some explicit examples of these conserved mixed currents. To that end, we need an explicit solution of the Einstein-Klein-Gordon equations. There are several of these in the literature —note that sometimes they are considered as "stiff fluid" solutions.— The problem, however, is that most of the solutions have either hypersurface-orthogonal Killing vectors, or spherical symmetry, or a G_2-group acting orthogonally transitively, so that the results of section 2 apply. This means that, for these particular solutions, the Bel and scalar-field s-e currents are, actually, divergence-free on their own. And thus, they do not give the *mixed* conservation we are seeking. What one needs, therefore, is a solution belonging to class A(ii) in Wainwright's classification of G_2 solutions [32, 31], that is to say, a solution with a non-orthogonally transitive G_2 group of motions.[4]

Remarkably and fortunately, there exists some explicitly known solutions for spacetimes belonging to class A(ii) and with a minimally coupled massless scalar field ϕ as source. These

[4] A spacetime with just one *non-hypersurface-orthogonal* Killing vector will also do, but we do not know of any such solutions for the Einstein-Klein-Gordon system.

were studied by Wils [34] who, among other metrics of the mentioned sort, found the one-parameter family of metrics that we are going to use here.

In general, the line element for such solutions will take the following form:

$$ds^2 = -F_0^2 dt^2 + F_1^2 dx^2 + F_2[F_3^2 dy^2 + F_3^{-2}(dz + F_4 dx)^2]$$

where F_0, F_1, F_2 and F_3 and F_4 are functions of t and x only. Wils found the mentioned exact solutions making separability assumptions on the functions involved, and the solution we are interested in has

$$\begin{aligned}
F_0(t,x) &= F_1(t,x) = \sinh^{2+\lambda} \frac{Lt}{\sqrt{6}}\, e^{\frac{(3+4\,\lambda)}{6} Lx}\,, \\
F_2(t,x) &= \sinh \frac{Lt}{\sqrt{6}}\, e^{\frac{\lambda}{3} Lx}\,, \\
F_3(t,x) &= \sinh^{-\frac{1+2\,\lambda}{2}} \frac{Lt}{\sqrt{6}}\, e^{-\frac{(3+\lambda)}{6} Lx}\,, \\
F_4(t,x) &= \cosh \frac{Lt}{\sqrt{6}} \sqrt{2 - \frac{4\,\lambda^2}{3}}\, e^{\frac{\lambda}{3} Lx}\,,
\end{aligned}$$

and

$$\phi = \sqrt{4 - 2\,\lambda^2}\, \log\left(\tanh \frac{Lt}{2\sqrt{6}}\right),$$

where L is a positive constant defining the scale and λ is a parameter subject to $\lambda^2 \leq 3/2$. The range of coordinates is constrained by the singularity at $t = 0$, and thus we take $t > 0$.

Obviously, $\vec{\xi} = \partial/\partial y$ is a Killing vector that satisfies $\boldsymbol{\xi} \wedge d\boldsymbol{\xi} = 0$, so that it is hypersurface orthogonal. From Result 2.1 it follows that the Bel current $\vec{j}(\vec{\xi})$ is parallel to $\vec{\xi}$ and hence conserved. In turn, and because the total mixed current $\vec{J}(\vec{\xi})$ is also conserved, the current $\vec{\mathcal{J}}(\vec{\xi})$ will be divergence-free too. Thus, the sought s-e interchange cannot be found here. We include the expression for $\vec{J}(\vec{\xi})$ for completeness:

$$\vec{J}(\vec{\xi}) = \frac{L^4}{72}\left(4(\lambda^3 - \lambda + 3)(2\lambda + 3)\sinh^2 \frac{Lt}{\sqrt{6}} - \lambda^4 + 20\lambda^2 + 72\lambda + 68\right) \sinh^{-6(2+\lambda)} \frac{Lt}{\sqrt{6}} e^{-\frac{(9+8\lambda)}{3} Lx}\, \vec{\xi}.$$

However, we can also use the second Killing vector, which is $\vec{\eta} = \partial/\partial z$, and one has

$$\boldsymbol{\eta} \wedge d\boldsymbol{\eta} = \frac{L}{3}\sqrt{3 - 2\lambda^2}\; e^{\frac{(6+5\,\lambda)}{3} Lx} \sinh^{5+4\,\lambda} \frac{Lt}{\sqrt{6}}\; dt \wedge dx \wedge dz,$$

meaning that $\vec{\eta}$ will only be hypersurface orthogonal in the extreme cases $\lambda = \pm\sqrt{3/2}$. Nevertheless, it can be checked that the current $\vec{j}(\vec{\xi}, \vec{\eta})$, as well as $\vec{\mathcal{J}}(\vec{\xi}, \vec{\eta})$, is in fact proportional to $\vec{\xi}$, so that both of them are divergence-free separately once more. Their sum reads

$$\begin{aligned}
\vec{J}(\vec{\xi}, \vec{\eta}) = &-\frac{L^4}{648}\left[3\left(\lambda^4 + 64\lambda^3 + 140\lambda^2 + 40\lambda - 44 - 2(3 - 2\lambda^2)^2\right)\sinh^4 \frac{Lt}{\sqrt{6}}\right. \\
&\left. -4\left(14\lambda^4 - 21\lambda^3 - 97\lambda^2 - 30\lambda + 36\right)\sinh^2 \frac{Lt}{\sqrt{6}}\right] \sinh^{-2(5+\lambda)} \frac{Lt}{\sqrt{6}} e^{-(1+2\lambda)Lx}\; \vec{\xi}.
\end{aligned}$$

The only left possibilities for a true exchange of s-e properties are to be found in the remaining conserved currents $\vec{J}(\vec{\eta})$ and $\vec{J}(\vec{\eta},\vec{\xi})$. And this is indeed the case. Setting $x^0=t$, $x^1=x$, $x^2=y$ and $x^3=z$, the expressions for their non-zero components are the following:

$$\begin{aligned}
J^0(\vec{\eta},\vec{\xi}) =&\frac{L^4}{324}(3+2\lambda)\sqrt{3-2\lambda^2}\left(4(1-2\lambda^2)+3\left(3-2\lambda^2\right)\sinh^2\frac{Lt}{\sqrt{6}}\right)\times\\
&\sinh^{-(11+6\lambda)}\frac{Lt}{\sqrt{6}}\,e^{-3(1+\lambda)Lx}\\
J^1(\vec{\eta},\vec{\xi}) =&-\frac{L^4}{54\sqrt{6}}\sqrt{3-2\lambda^2}\left(2\left(3+2\lambda\right)\left(1-2\lambda^2\right)+3\left(1+\lambda\right)\left(3-2\lambda^2\right)\sinh^2\frac{Lt}{\sqrt{6}}\right)\times\\
&\cosh\frac{Lt}{\sqrt{6}}\sinh^{-6(2+\lambda)}\frac{Lt}{\sqrt{6}}\,e^{-3(1+\lambda)Lx}\\
J^3(\vec{\eta},\vec{\xi}) =&\frac{L^4}{648}\Big(204-72\lambda-612\lambda^2-320\lambda^3+93\lambda^4+64\lambda^5+4\left(81+9\lambda-181\lambda^2-\right.\\
&\left.-89\lambda^3+50\lambda^4+28\lambda^5\right)\sinh^2\frac{Lt}{\sqrt{6}}+6\,(3+2\lambda)\left(3-2\lambda^2\right)^2\sinh^4\frac{Lt}{\sqrt{6}}\Big)\times\\
&\sinh^{-6(2+\lambda)}\frac{Lt}{\sqrt{6}}\,e^{-\frac{(9+8\lambda)}{3}Lx}\\
J^0(\vec{\eta}) =&\frac{L^4}{54}\lambda\sqrt{3-2\lambda^2}\left(1-\left(3-2\lambda^2\right)\sinh^2\frac{Lt}{\sqrt{6}}\right)\sinh^{-9-2\lambda}\frac{Lt}{\sqrt{6}}\,e^{-\frac{(3+7\lambda)Lx}{3}}\\
J^1(\vec{\eta}) =&-\frac{L^4}{18\sqrt{6}}\sqrt{3-2\lambda^2}\left(2-\left(3-2\lambda^2\right)\sinh^2\frac{Lt}{\sqrt{6}}\right)\cosh\frac{Lt}{\sqrt{6}}\sinh^{-2(5+\lambda)}\frac{Lt}{\sqrt{6}}\times\\
&e^{-\frac{(3+7\lambda)}{3}Lx}\\
J^3(\vec{\eta}) =&\frac{L^4}{216}\Big(228+216\,\lambda+44\,\lambda^2-3\,\lambda^4+4\,\left(18+9\,\lambda+6\,\lambda^2+9\,\lambda^3+2\,\lambda^4\right)\sinh^2\frac{Lt}{\sqrt{6}}-\\
&-2\left(3-2\,\lambda^2\right)^2\sinh^4\frac{Lt}{\sqrt{6}}\Big)\sinh^{-2(5+\lambda)}\frac{Lt}{\sqrt{6}}\,e^{-(1+2\,\lambda)\,Lx}.
\end{aligned}$$

It can be checked by an explicit computation that these vector fields are divergence-free. We want to stress that the corresponding pure currents $\vec{j}(\vec{\eta})$ and $\vec{\mathfrak{J}}(\vec{\eta})$, or $\vec{j}(\vec{\eta},\vec{\xi})$ and $\vec{\mathfrak{J}}(\vec{\eta},\vec{\xi})$, are *not* divergence-free in general (unless, of course, $3-2\lambda^2=0$ in which case $\{\vec{\xi},\vec{\eta}\}$ generate an orthogonal transitive G_2, in agreement with our previous results). More importantly, we must remark that these currents $\vec{J}(\vec{\eta})$ and $\vec{J}(\vec{\eta},\vec{\xi})$ are *non-trivial* in the sense that they are *not* linear combinations of the Killing vectors. In other words, they would be very difficult to find if we did not know about the superenergy concept. The explicit expressions for superenergy tensors of the gravitational and scalar fields are essential here.

A door obviously open by our work can be stated in the form of the following yet unanswered question: are there similar non-trivial *mixed* superenergy currents for general Einstein-Maxwell systems?

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A Appendix

The 2-form associated with the two Killing vectors spanning the orbits will be denoted by

$$\boldsymbol{w} \equiv \boldsymbol{\xi} \wedge \boldsymbol{\eta}. \tag{33}$$

We also define the one-form $\boldsymbol{r}\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_C}\right)$, relative to the vector fields $\vec{\xi}_A,\vec{\xi}_B,\vec{\xi}_C$, whose components are given by

$$r_\mu\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_C}\right) \equiv \xi_A^\lambda\,\xi_B^\alpha\,R_\alpha{}^\rho{}_\lambda{}^\sigma\xi_C^\beta R_{\beta\rho\sigma\mu}, \tag{34}$$

where we have introduced the indices $A,B,C=1,2$ to denote the two independent Killing vectors generating the G_2 group: $\vec{\xi}_1 \equiv \vec{\xi}$ and $\vec{\xi}_2 \equiv \vec{\eta}$. It is convenient to define $\boldsymbol{t}\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_C}\right)$ as

$$t_\mu\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_C}\right) \equiv \xi_A^\alpha \eta_{\alpha\nu\rho_1\rho_2}\xi_{B\beta}\eta^{\beta\nu\alpha_1\alpha_2}\xi_C^\lambda R_\lambda{}^{\sigma\rho_1\rho_2}R_{\alpha_1\alpha_2\sigma\mu}, \tag{35}$$

where $\eta_{\alpha\beta\mu\nu}$ is the volume element, so that using (2), (34) and (35), a straightforward calculation allows us to get

$$\begin{aligned}\boldsymbol{j}\left(\vec{\xi_B},\vec{\xi_A}\right)=&\frac{1}{3}\Bigg\{-\boldsymbol{r}\left(\vec{\xi_B},\vec{\xi_A},\vec{\xi_A}\right)-2\boldsymbol{r}\left(\vec{\xi_A},\vec{\xi_A},\vec{\xi_B}\right)-\frac{1}{4}\boldsymbol{t}\left(\vec{\xi_A},\vec{\xi_A},\vec{\xi_B}\right)-\frac{1}{2}\boldsymbol{t}\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_A}\right)-\\ &-\frac{1}{2}\xi_A^\alpha R_{\alpha\beta\lambda\sigma}\;\xi_A^\rho R_\rho{}^{\beta\lambda\sigma}\;\boldsymbol{\xi}_B-\xi_B^\alpha R_{\alpha\beta\lambda\sigma}\;\xi_A^\rho R_\rho{}^{\beta\lambda\sigma}\;\boldsymbol{\xi}_A+\\ &+\frac{1}{8}(\vec{\xi}_A\cdot\vec{\xi}_A)R^{\rho\beta\lambda\sigma}R_{\rho\beta\lambda\sigma}\;\boldsymbol{\xi}_B+\frac{2}{8}(\vec{\xi}_A\cdot\vec{\xi}_B)R^{\rho\beta\lambda\sigma}R_{\rho\beta\lambda\sigma}\;\boldsymbol{\xi}_A\Bigg\}.\end{aligned}$$

Clearly the exterior product $\boldsymbol{j}\wedge\boldsymbol{w}$ contains only terms of the form $\boldsymbol{r}\wedge\boldsymbol{w}$ and $\boldsymbol{t}\wedge\boldsymbol{w}$. We define now the 1-forms $\boldsymbol{\Sigma}_A$ and $\boldsymbol{\Omega}_A$ by

$$\begin{aligned}\Sigma_{A\mu} &\equiv \xi^\alpha\eta^\beta\,\xi_A^\lambda\,R_{\alpha\beta\lambda\mu},\\ \Omega_{A\mu} &\equiv *w^{\alpha\beta}\,\xi_A^\lambda\,R_{\alpha\beta\lambda\mu}.\end{aligned}$$

Given that $\vec{\xi}_A$ are Killing vectors, we have

$$\xi^\sigma\eta^\rho R_{\sigma\rho\alpha\beta}=\nabla_\alpha[\vec{\eta},\vec{\xi}]_\beta+2\nabla_\rho\eta_{[\alpha}\nabla^\rho\xi_{\alpha]}.$$

Contracting this identity with a third Killing vector, and making the exterior product with $\boldsymbol{w}$, it is straighforward to show

$$\Sigma_{A[\alpha}w_{\mu\nu]} = 2\xi_A^\beta\nabla^\rho\eta_\beta\, w_{[\mu\nu}\xi_{\alpha;\rho]} - 2\xi_A^\beta\nabla^\rho\xi_\beta\, w_{[\mu\nu}\eta_{\alpha;\rho]} + \xi_A^\beta\nabla_\beta[\vec{\eta},\vec{\xi}]_{[\alpha}w_{\mu\nu]} + \frac{1}{3!}\,\vec{\xi}_A\cdot[\vec{\eta},\vec{\xi}]\,\mathrm{d}\boldsymbol{w}_{\alpha\mu\nu} \quad (36)$$

Using (8) it is easy to show that

$$\begin{aligned}\Omega_{A[\alpha}w_{\mu\nu]} =& \frac{1}{3!}\left(\mathrm{d}*(\boldsymbol{w}\wedge\mathrm{d}\boldsymbol{\xi}_A)\wedge\boldsymbol{w} - *(\boldsymbol{w}\wedge\mathrm{d}\boldsymbol{\xi}_A)\,\mathrm{d}\boldsymbol{w}\right)_{\alpha\mu\nu}\\ &-2*(\boldsymbol{\eta}\wedge\mathrm{d}\boldsymbol{\xi}_A)^\rho w_{[\rho\alpha}\xi_{\nu;\mu]} + 2*(\boldsymbol{\xi}\wedge\mathrm{d}\boldsymbol{\xi}_A)^\rho w_{[\rho\alpha}\eta_{\nu;\mu]}\,. \end{aligned} \quad (37)$$

Bearing in mind these identities, the following definitions will be useful later:

$$\tilde{\Sigma}_{A\alpha} \equiv \frac{1}{4}\Sigma_{A[\alpha}w_{\mu\nu]}w^{\mu\nu} = -\frac{1}{3!}\Sigma_{A\sigma}*w^{\sigma\epsilon}*w_{\alpha\epsilon}, \quad (38)$$

$$\tilde{\Omega}_{A\alpha} \equiv \frac{1}{4}\Omega_{A[\alpha}w_{\mu\nu]}w^{\mu\nu} = -\frac{1}{3!}\Omega_{A\sigma}*w^{\sigma\epsilon}*w_{\alpha\epsilon}. \quad (39)$$

In order to obtain an expression for $\boldsymbol{r}\wedge\boldsymbol{w}$ in terms of the above quantities we will use the decomposition of the identity $\delta^\alpha_\beta = P^\alpha_{\perp\beta} + P^\alpha_{\parallel\beta}$ into its part on the orbit and the corresponding orthogonal element at each point, i.e.

$$P^\alpha_{\perp\beta} = -\frac{1}{W}*w^{\alpha\rho}*w_{\beta\rho}, \quad P^\alpha_{\parallel\beta} = \frac{1}{W}w^{\alpha\rho}w_{\beta\rho}, \quad W = 2w^{\alpha\beta}w_{\alpha\beta} = (\vec{\xi}\cdot\vec{\xi})(\vec{\eta}\cdot\vec{\eta}) - (\vec{\xi}\cdot\vec{\eta})^2. \quad (40)$$

Furthermore, it is convenient to compute first $*(\boldsymbol{r}\wedge\boldsymbol{w})$ and apply the Hodge dual to the final identity. One can then write

$$*(\boldsymbol{r}\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_C}\right)\wedge\boldsymbol{w})^\gamma = \xi_A^\lambda\,\xi_B^\mu\,R_\mu{}^\rho{}_\lambda{}^\sigma(P^\tau_{\perp\rho}+P^\tau_{\parallel\rho})(P^\epsilon_{\perp\sigma}+P^\epsilon_{\parallel\sigma})\xi_C^\beta R_{\beta\tau\epsilon\alpha}*w^{\alpha\gamma}.$$

Taking into account that $P^\epsilon_{\perp\rho}z_{\epsilon\alpha}*w^{\alpha\gamma} = \frac{1}{2}P^\gamma_{\perp\rho}z_{\epsilon\alpha}*w^{\alpha\epsilon}$ for any 2-form $\boldsymbol{z}$, this identity can be arranged onto its dual form as follows

$$\begin{aligned} r_{[\alpha}\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_C}\right)w_{\mu\nu]} = \frac{1}{W^2}\Big\{&-6R_B{}^\rho{}_A{}^\sigma\tilde{\Omega}_{C\rho}*w_{\sigma[\alpha}w_{\mu\nu]}\\ &-(\vec{\xi_B}\cdot\vec{\xi_C})*w^{\lambda\sigma}R_{12\lambda\sigma}(*\tilde{\boldsymbol{\Sigma}}_A)_{\alpha\mu\nu} - 6\,\xi_{A\gamma}w^{\epsilon\gamma}\tilde{\Sigma}_{B\delta}R_C{}^\delta{}_{\epsilon[\alpha}w_{\mu\nu]}\\ &-(\vec{\xi_B}\cdot\vec{\xi_C})R_{1212}\,\xi_A^\lambda\left(\eta_\lambda\Sigma_{1[\alpha}w_{\mu\nu]} - \xi_\lambda\Sigma_{2[\alpha}w_{\mu\nu]}\right)\Big\}, \end{aligned} \quad (41)$$

where we have used the indices $A, B, C, 1, 2$ in some places also to denote contractions with $\vec{\xi}_A, \vec{\xi}_B, \vec{\xi}_C, \vec{\xi}, \vec{\eta}$ respectively, in an obvious manner.

Let us proceed now with $\boldsymbol{t}\wedge\boldsymbol{w}$. Using the previous decomposition of δ^α_β for the summation index ν in (35) and arranging terms conveniently, we get

$$\begin{aligned} t_{[\alpha}\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_C}\right)w_{\mu\nu]} = \frac{1}{W}\Big\{&-4\,\xi_A^\lambda w_{\rho_2\lambda}\xi_{B\beta}w^{\alpha_2\beta}R_C{}^{\sigma\rho_1\rho_2}R_{\rho_1\alpha_2\sigma[\alpha}w_{\mu\nu]}\\ &+(\vec{\xi}_A\cdot\vec{\xi}_B)w^{\rho_1\rho_2}\xi_C^\lambda R_{\rho_1\rho_2\lambda}{}^\sigma w^{\alpha_1\alpha_2}R_{\alpha_1\alpha_2\sigma[\alpha}w_{\mu\nu]}\\ &+(\vec{\xi}_A\cdot\vec{\xi}_B)*w^{\rho_1\rho_2}\xi_C R_{\rho_1\rho_2\lambda}{}^\sigma*w^{\alpha_1\alpha_2}R_{\alpha_1\alpha_2\sigma[\alpha}w_{\mu\nu]}\Big\}. \end{aligned}$$

The first term has the form $r_{[\alpha}\left(\vec{\xi_C},\vec{v_A},\vec{v_B}\right)w_{\mu\nu]}$ with $v_A{}^\rho=\xi_{A\lambda}w^{\rho\lambda}$, and hence can be expanded using (41). For the second and third terms we still have to apply the same procedure as above to the summation index σ. After some calculation we can obtain the following expression

$$\begin{aligned}\frac{1}{4}t_{[\alpha}\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_C}\right)w_{\mu\nu]}=&\frac{3!}{W^3}\Big[\xi_B^\lambda(\eta_\lambda\tilde{\Omega}_{1\rho}-\xi_\lambda\tilde{\Omega}_{2\rho})\xi_{A\gamma}w^{\tau\gamma}\xi_C^\epsilon R_\tau{}^\rho{}_\epsilon{}^\sigma*w_{\sigma[\alpha}w_{\mu\nu]}\\&+\xi_A^\lambda(\eta_\lambda\tilde{\Sigma}_{1\rho}-\xi_\lambda\tilde{\Sigma}_{2\rho})\xi_{B\gamma}w^{\tau\gamma}\xi_{C\delta}w^{\epsilon\delta}R_\tau{}^\rho{}_{\epsilon[\alpha}w_{\mu\nu]}\Big]+\\&+\frac{1}{4W^2}(\vec{\xi}_A\cdot\vec{\xi}_B)\Big\{3!(2\tilde{\Sigma}_C{}^\tau w^{\lambda\sigma}+\tilde{\Omega}_C{}^\tau*w^{\lambda\sigma})R_{\lambda\sigma\tau[\alpha}w_{\mu\nu]}\\&+[4(*\tilde{\Sigma}_C)_{\alpha\mu\nu}-\xi_C^\epsilon(\eta_\epsilon\Sigma_{1[\alpha}w_{\mu\nu]}-\xi_\epsilon\Sigma_{2[\alpha}w_{\mu\nu]})]*w^{\lambda\sigma}R_{12\,\lambda\sigma}\Big\}.\end{aligned}\quad(42)$$

Using (41) and (42) we are now ready to obtain the expressions for

$$\begin{aligned}\boldsymbol{j}\left(\vec{\xi_B},\vec{\xi_A}\right)\wedge\boldsymbol{w}=-\frac{1}{3}\Big[&\boldsymbol{r}\left(\vec{\xi_B},\vec{\xi_A},\vec{\xi_A}\right)\wedge\boldsymbol{w}+2\boldsymbol{r}\left(\vec{\xi_A},\vec{\xi_A},\vec{\xi_B}\right)\wedge\boldsymbol{w}\\&+\frac{1}{4}\boldsymbol{t}\left(\vec{\xi_A},\vec{\xi_A},\vec{\xi_B}\right)\wedge\boldsymbol{w}+\frac{1}{2}\boldsymbol{t}\left(\vec{\xi_A},\vec{\xi_B},\vec{\xi_A}\right)\wedge\boldsymbol{w}\Big],\end{aligned}$$

which, after using $\xi_{A\gamma}w^{\lambda\gamma}(\eta_\lambda\tilde{\Omega}_{1\rho}-\xi_\lambda\tilde{\Omega}_{2\rho})=-W\tilde{\Omega}_{A\rho}$ and analogous identities, can be explicitly written as (14).

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